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by

SHELDON M. ROSS, MEHRDAD SHAHSHAHANI and GIDEON WEISS

OPERATIONS RESEARCH CENTER

UNIVERSITY OF CALIFORNIA · BERKELEY

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ON THE NUMBER OF COMPONENT FAILURES IN SYSTEM WHOSE COMPONENT LIVES ARE EXCHANGEABLE

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OCTOBER 1978 ORC 78-19

This research has been partially supported by the Air Force Office of Scientific Research (AFSC), USAF, under Grant AFOSR-77-3213 and the Office of Naval Research under Contract N00014-77-C-0299 with the University of California. Reproduction in whole or in part is permitted for any purpose of the United States Government.

Unclassified.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER ORC-78-19	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) ON THE NUMBER OF COMPONENT FAILURES IN SYSTEM WHOSE COMPONENT LIVES ARE EXCHANGEABLE.		5. TYPE OF REPORT & PERIOD COVERED Research Report.
		6. PERFORMING ORG. REPORT NUMBER N00014-77-C-0299
7. AUTHOR(s) Sheldon M. Ross, Mehrdad Shahshahani and Gideon Weiss		8. CONTRACT OR GRANT NUMBER(s) AFOSR-77-3213
9. PERFORMING ORGANIZATION NAME AND ADDRESS Operations Research Center University of California Berkeley, California 94720		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 2304/A5
11. CONTROLLING OFFICE NAME AND ADDRESS United States Air Force Air Force Office of Scientific Research Bolling AFB, D.C. 20332		12. REPORT DATE October 1978
		13. NUMBER OF PAGES 19
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office) 24p.		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

Also supported by the Office of Naval Research under Contract N00014-77-C-0299.

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Monotone System
Increasing Failure Rate Average
Increasing Failure Rate
Minimal Cut Sets
Minimal Path Sets
Factorial Moments

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

(SEE ABSTRACT)

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE
S/N 0102-LF-014-6601

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

270 750

ABSTRACT

We consider the usual n component monotone system in which each component is either "on" or "off" at any time. That is, letting

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is on} \\ 0 & \text{otherwise} \end{cases}$$

and $\underline{x} = (x_1, \ldots, x_n)$, we suppose that there is a nondecreasing function ϕ such that

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if the system is on under state vector } \underline{x} \\ 0 & \text{otherwise.} \end{cases}$$

The function ϕ is called the structure function.

Consider now an arbitrary such system and suppose that the i^{th} component is initially on and stay on for a random time T_i of which point it goes off and remains off forever. The random times T_i, $i = 1, \ldots, n$ will be assumed to be independent and identically distributed continuous random variables. We are interested in studying the properties of N, the number of components that are off at the moment the system goes off. In Section 1 we compute the factorial moments of N in terms of the reliability function. In Section 2 we prove that N is an increasing failure rate average random variable and we also present a duality result. In Section 3 we consider the special structure in which the minimal cut sets do not overlap and we prove a conjecture of El-Neweihi, Proschan and Sethuraman which states that N is an increasing failure rate random variable. In the final section we consider the special case of nonoverlapping min path sets.

ON THE NUMBER OF COMPONENT FAILURES IN SYSTEMS
WHOSE COMPONENT LIVES ARE EXCHANGEABLE

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0. INTRODUCTION AND SUMMARY

We consider the usual n component monotone system in which each component is either "on" or "off" at any time. That is, letting

$$x_i = \begin{cases} 1 & \text{if component } i \text{ is on} \\ 0 & \text{otherwise} \end{cases}$$

and $\underline{x} = (x_1, \ldots, x_n)$, we suppose that there is a nondecreasing function ϕ such that

$$\phi(\underline{x}) = \begin{cases} 1 & \text{if the system is on under state vector } \underline{x} \\ 0 & \text{otherwise.} \end{cases}$$

The function ϕ is called the structure function. If X_i , $i = 1, \ldots, n$ are assumed to be independent binary random variables with $P\{X_i = 1\} = p_i = 1 - P\{X_i = 0\}$ then we define the reliability function $r(\underline{p})$ by

$$r(\underline{p}) = P\{\phi(\underline{X}) = 1\} = E[\phi(\underline{X})] \ .$$

Consider now an arbitrary such system and suppose that the i^{th} component is initially on and stay on for a random time T_i of which point it goes off and remains off forever. The random times T_i , $i = 1, \ldots, n$ will be assumed to be independent and identically distributed continuous random variables (though as can easily be seen all of our results will only depend on the fact that their joint distribution is exchangeable).

We are interested in studying the properties of N, the number of components that are off at the moment the system goes off. (That is, N is the number of component failures necessary to cause system failure under the assumption that every time a component failure occurs the failed component is equally likely to be any of the components that were up at that time.) In Section 1 we compute the factorial moments of N in terms of the reliability function. In Section 2 we prove that N is an increasing failure rate average random variable and we also present a duality result. In Section 3 we consider the special structure in which the minimal cut sets do not overlap and we prove a conjecture of El-Neweihi, Proschan and Sethuraman [2] which states that N is an increasing failure rate random variable. In the final section we consider the special case of nonoverlapping min path sets. In this section we make extensive use of the duality principle to extend many of the earlier results both of this paper and of [2].

1. FACTORIAL MOMENTS OF N

Let F denote the distribution of component "on" time and let $\bar{F} = 1 - F$. If we let T denote the time at which the system goes off then

$$T = T_{(N)}$$

where $T_{(1)} \leq T_{(2)} \cdots \leq T_{(n)}$ are the order statistics of $T_1, \ldots, T_n$. Now let $Y_1, \ldots, Y_k$ denote k independent (of each other and also of the T_i) random variables each having distribution F, and consider $P\left\{\max_{j=1,\ldots,k} Y_j < T_{(N)}\right\}$. We have

$$P\left\{\max_{1 \leq j \leq k} Y_j < T_{(N)}\right\} = \sum_{i=1}^{n} P\{\max Y_j < T_{(i)} \mid N = i\}P\{N = i\}$$

(1)

$$= \sum_{i=1}^{n} P\{\max Y_j < T_{(i)}\}P\{N = i\}$$

where the last equality follows from the fact that knowing that $N = i$ gives us information about the identity of the components which fail but by the symmetry (exchangeable) assumption this yields no information about the times at which these failures occurred. Now, $P\left\{\max_{1 \leq j \leq k} Y_j < T_{(i)}\right\}$ is just the probability that a given set of k elements in a set of $n + k$ elements are all chosen within the first $k + i - 1$ selections in a nonreplacement random selection scheme. Thus

$$P\left\{\max_{1\le j\le k} Y_j < T_{(i)}\right\} = \frac{\binom{n}{i-1}}{\binom{n+k}{k+i-1}} = \frac{n!}{(n+k)!}\,\frac{(k+i-1)!}{(i-1)!}\ .$$

Substituting in (1) yields

$$\text{(2)}\quad P\left\{\max_{1\le j\le k} Y_j < T_{(N)}\right\} = \frac{n!}{(n+k)!}\,E[N(N+1)\cdots(N+k-1)]\ .$$

However we can also obtain an expression for $P\{\max Y_j < T_{(N)}\}$ by conditioning on $\max Y_j$ as follows:

$$P\left\{\max_{1\le j\le k} Y_j < T_{(N)}\right\} = \int_0^\infty P\{T_{(N)} > t\}kF^{k-1}(t)dF(t)$$

$$\text{(3)}\qquad = \int_0^\infty r(\bar{F}(t),\ \ldots,\ \bar{F}(t))kF^{k-1}(t)dF(t)$$

$$= k\int_0^1 r(p,\ \ldots,\ p)(1-p)^{k-1}dp$$

where we have used the well-known fact (see [1] or [3]) that $P\{T > t\} = r(\bar{F}(t),\ \ldots,\ \bar{F}(t))$. Equating (2) and (3) yields

<u>Theorem 1</u>:

$$E[N(N+1)\cdots(N+k-1)] = \frac{(n+k)!}{n!}\,k\int_0^1 r(p)(1-p)^{k-1}dp$$

where $r(p) = r(p,p,\ \ldots,\ p)$.

2. INCREASING FAILURE RATE AVERAGE (IFRA) AND DUALITY

The failure rate of a discrete positive random variable X is defined as:

$$\lambda_k = \frac{P(X = k)}{P(X > k - 1)}, \quad k = 1,2, \ldots .$$

X is called increasing failure rate (IFR) if λ_k is nondecreasing in k. X is called IFRA if $\frac{1}{k} \sum_{j=1}^{k} \lambda_j$ is nondecreasing in k. We say X is SSLSF (has star-shaped log survival function) if $\{P(X > k)\}^{1/k}$ is nonincreasing in k. For continuous r.v. IFR $\Rightarrow$ SSLSF $\Leftrightarrow$ IFRA ; in the discrete case we have:

Lemma 1:

IFR $\Rightarrow$ SSLSF $\Rightarrow$ IFRA .

Proof:

We show first that:

$$P(X > k) = \prod_{j=1}^{k} (1 - \lambda_j) .$$

To see that we note that $P(X > 1) = 1 - P(X = 1) = 1 - \lambda_1$, and proceed by induction to write:

$$P(X > k + 1) = P(X > k) - P(X = k + 1)$$

$$= P(X > k) - \lambda_{k+1} P(X > k) = \prod_{j=1}^{k+1} (1 - \lambda_j) .$$

Now X is (a) IFR, (b) SSLSF or (c) IFRA respectively iff (a) $1 - \lambda_k$, (b) $\left\{\prod_{j=1}^{k} (1 - \lambda_j)\right\}^{1/k}$, or (c) $\frac{1}{k} \sum_{j=1}^{k} (1 - \lambda_j)$ is nonincreasing in k; equivalently iff for $k = 1,2, \ldots, 1 - \lambda_{k+1}$ is less than or equal to (a) $1 - \lambda_k$, (b) $\left\{\prod_{j=1}^{k} (1 - \lambda_j)\right\}^{1/k}$, or (c) $\frac{1}{k} \sum_{j=1}^{k} (1 - \lambda_j)$. Hence the lemma follows by comparison of the smallest term, the geometric mean, and the arithmetic mean.■

In addition to the random variable N we shall look at random variables $N(0_i)$ and $N(1_i)$, $i = 1, \ldots, n$, which will be the number of component failures until system failure in the $n - 1$ component systems with respective structure functions $\phi(0_i;\underline{x})$ and $\phi(1_i;\underline{x})^{\dagger}$.

By conditioning on the first component failure, or by conditioning on whether component i fails among the first k or not we get for $i = 1, \ldots, n$:

$$P(N > k) = \sum_{j=1}^{n} \frac{1}{n} P(N(0_j) > k - 1)$$

(4)

$$= \frac{k}{n} P(N(0_i) > k - 1) + \frac{n - k}{n} P(N(1_i) > k) .$$

We use these two different representations to see that for every k there exists an i for which:

(5) $$P(N(0_i) > k) \leq P(N(1_i) > k + 1) .$$

In the following proof we shall use the inequality (see [3], page 217) for $0 \leq \alpha \leq 1$, $0 \leq \lambda \leq 1$, $0 \leq y \leq x$

$^{\dagger}\phi(0_i,\underline{x}) = \phi(x_1, \ldots, x_{i-1},0,x_{i-1}, \ldots, x_n)$

$\phi(1_i,\underline{x}) = \phi(x_1, \ldots, x_{i-1},1,x_{i+1}, \ldots, x_n)$.

(6) $$\lambda^\alpha x^\alpha + (1 - \lambda^\alpha)y^\alpha \geq (\lambda x + (1 - \lambda)y)^\alpha .$$

We shall also use the fact that, for $a \geq 1$, $a \geq x \geq 0$:

(7) $$(a - x)^{1/x} \text{ is decreasing in } x$$

which is easily seen by taking logarithms and writing

$$\frac{1}{x} \log (a - x) = \frac{\log a}{x} - \sum_{m=1}^{\infty} \frac{x^{m-1}}{ma^m} .$$

We now prove:

__Theorem 2__:

N is IFRA.

__Proof__:

We will show by induction on n that N has the stronger property of SSLSF. For $n = 1$, $N = 1$ and N is IFRA. We shall now show that

$$\{P(N > k + 1)\}^{\frac{1}{k+1}} \leq \{P(N > k)\}^{\frac{1}{k}}$$

while assuming that $N(0_i)$ and $N(1_i)$ are SSLSF; we choose i here to satisfy (5).

We start by using (4) to write:

$$P(N > k) = \frac{k}{n} P(N(0_i) > k - 1) + \frac{n - k}{n} P(N(1_i) > k) .$$

By the induction hypothesis the above is

$$\geq \frac{k}{n} P(N(0_i) > k)^{\frac{k-1}{k}} + \frac{n-k}{n} P(N(1_i) > k+1)^{\frac{k}{k+1}} .$$

Since $\frac{k-1}{k} < \frac{k}{k+1}$ and $P(N(0_i) > k) \leq 1$ the above set of inequalities can be continued as follows:

$$\geq \frac{k}{n} P(N(0_i) > k)^{\frac{k}{k+1}} + \frac{n-k}{n} P(N(1_i) > k+1)^{\frac{k}{k+1}} .$$

Now, by the inequality (6) applied to $\alpha = \frac{k}{k+1}$,

$$x = P(N(1_i) > k+1) \geq P(N(0_i) > k) = y ,$$

and

$$\lambda = \left(\frac{n-k}{n}\right)^{\frac{k+1}{k}}$$

we have that the above is

$$\geq \left\{\left(1 - \left(\frac{n-k}{n}\right)^{\frac{k+1}{k}}\right) P(N(0_i) > k) + \left(\frac{n-k}{n}\right)^{\frac{k+1}{k}} P(N(1_i) > k+1)\right\}^{\frac{k}{k+1}} .$$

Finally by (5) and by the fact that from (7)

$$\left(\frac{n-k}{n}\right)^{\frac{1}{k}} > \left(\frac{n-k-1}{n}\right)^{\frac{1}{k+1}} ,$$

we see that the above is

$$\geq \left\{ \frac{k+1}{n} P(N(0_i) > k) + \frac{n-k-1}{n} P(N(1_i) > k+1) \right\}^{\frac{k}{k+1}}$$

$$= \{P(N > k+1)\}^{\frac{k}{k+1}} . \blacksquare$$

A cut set is a set of components whose failure (of all components in the set) ensures system failure. If a cut set has no proper subset which is also a cut set then it is said to be a minimal cut set. Similarly a minimal path set is a minimal (in the same sense as above) set of components whose functioning ensures the system's functioning.

Hence if we sample components one at a time, without replacement, then N is the number of components that must be sampled until the set of sampled components contains all the components of at least one minimal cut set, or, equivalently, at least one component of each minimal path set.

Let us denote by N^* the number of components that would have to be sampled until we have sampled all of the components of at least one minimal *path* set. Now as each of the $n!$ outcomes of the order in which the n components are sampled are assumed to be equally likely we can associate with any particular outcome - say $(i_1, i_2, \ldots, i_n)$, the outcome (having equal probability of occurring) which reverses the sampling order - that is $(i_n, i_{n-1}, \ldots, i_2, i_1)$. Now it is easy to see that

$$(i_1, \ldots, i_n) \Rightarrow N = k \Leftrightarrow (i_n, \ldots, i_1) \Rightarrow N^* = n - k + 1$$

for if the first k components $i_1, \ldots, i_k$ contain a minimal cut set and none of the first j components $i_1, \ldots, i_j$, $j < k$, do, then it follows that components $i_{k+1}, \ldots, i_n$ does not contain a minimal path

set but $i_{j+1}, \ldots, i_n$ does for all $j < k$.

Hence by the duality of outcomes $(i_1, \ldots, i_n)$ and $(i_n, \ldots, i_1)$ we see that

$$P\{N = k\} = P\{N^* = n - k + 1\} .$$

Since N^* is an increasing failure rate average random variable by exactly the same result as established this fact for N we have that

$$(P\{N^* > i\})^{1/i} \text{ is nonincreasing in } i$$

or, equivalently,

Corollary 1:

$$(P\{N < n + 1 - i\})^{1/i} \text{ is nonincreasing in } i .$$

3. NONOVERLAPPING MINIMAL CUT SETS

In this section we suppose that the minimal cut sets have no components in common. (In reliability terms such a system would be a series arrangement of nonoverlapping parallel structures.) This case has previously been studied in [2] where among other things, it was conjectured that N is an increasing failure rate (IFR) random variable. We now prove this.

Theorem 3:

If the minimal cut sets do not overlap then N is an IFR random variable in the sense that

$$P\{N = k + 1 \mid N > k\} \text{ is nondecreasing in } k .$$

Proof:

Let the cut sets be numbered $1, \ldots, r$, with $n_1, \ldots, n_r$ components respectively. Let D_k be the index of the cut set in which the k^{th} component failure has occurred $\Big(k = 1,2, \ldots, n$ where $n = \sum_{i=1}^{r} n_i$, where we assume that component failures continue to happen irrespectively of system failure$\Big)$. Let $x_j(k)$ be the number of components from cut set j that have failed in the first k component failures.

We prove that N is IFR by induction on r. For $r = 1$, $N = n_1$ is constant and hence IFR. We now assume the theorem for $r - 1$, and in particular we assume that N' the number of component failures until the system composed of cut sets $2, \ldots, r$ with $n_2, \ldots, n_r$ fails is IFR.

We note first that:

$$P(N = k + 1 \mid N > k) = \sum_{j=1}^{r} P(D_{k+1} = j \;,\; x_j(k) = n_j - 1 \mid N > k)$$

$$= \sum_{j=1}^{r} \frac{1}{n - k} P(x_j(k) = n_j - 1 \mid N > k)$$

since $P(D_{k+1} = j \mid x_j(k) = n_j - 1 \;,\; N > k)$ is independent of the event $N > k$ and is just the probability that the last component of set j should fail.

Since $\frac{1}{n - k}$ is increasing in k, it is enough to show that for all j, $P(x_j(k) = n_j - 1 \mid N > k)$ is nondecreasing in k. Obviously it is enough to look at $j = 1$, and as $P(x_1(k) = n_1 - 1 \mid N > k) = 0$ for $k = 0,1, \ldots, n_1 - 2$, we need only consider $k \geq n_1 - 1$.

We use the definition of N' to write

$$P(x_1(k) = n_1 - 1 \mid N > k) = \frac{P(x_1(k) = n_1 - 1 \;,\; N > k)}{\sum_{i=0}^{n_1-1} P(x_1(k) = i \;,\; N > k)}$$

$$= \frac{P(x_1(k) = n_1 - 1)P(N > k \mid x_1(k) = n_1 - 1)}{\sum_{i=0}^{n_1-1} P(x_1(k) = i)P(N > k) \mid x_1(k) = i)} = \frac{P(x_1(k) = n_1 - 1)P(N' > k - n_1 + 1)}{\sum_{i=0}^{n_1-1} P(x_1(k) = i)P(N' > k - i)} .$$

To show that this is nondecreasing it is enough to show that for $0 \leq i \leq n_1 - 1 \leq k$

$$\frac{P(x_1(k) = i)P(N' > k - i)}{P(x_1(k) = n_1 - 1)P(N' > k - n_1 + 1)}$$

is nonincreasing in k.

The assumption that N' is IFR implies that $P(N' > k - i)/P(N' > k - n_1 + 1)$ is nonincreasing in k. Finally,

$$\frac{P(x_1(k) = i)}{P(x_1(k) = n_1 - 1)} = \frac{\binom{n_1}{i}\binom{n - n_1}{k - i}}{\binom{n}{k}} \div \frac{\binom{n_1}{n_1 - 1}\binom{n - n_1}{k - n_1 + 1}}{\binom{n}{k}}$$

$$= \frac{(n_1 - 1)!}{i!(n_1 - i)!} \frac{(k - n_1 + 1)!(n - n_1 - k + n_1 - 1)!}{(k - i)!(n - n_1 - k + i)!}$$

and this expression is easily seen to be nonincreasing in k. ■

Remark:

The same proof, with an obvious modification, would suffice to show that N is IFR if system failure occurs the first time that r_i components from the i^{th} minimal cut set fail, for some i, where $r_i \leq n_i$. (Such a system would be a series arrangement of nonoverlapping r_i of n_i structures.)

4. NONOVERLAPPING MINIMAL PATH SETS

There is no analogue to Theorem 3 in the case where the minimal path sets do not overlap (such a system would be a parallel arrangement of nonoverlapping series structures). For instance if $n = 10$ and 1 is a minimal path set and components $2, \ldots, 10$ consist of a second minimal path set then

$$P\{N = k + 1 \mid N > k\} = \begin{cases} 0 & k = 0 \\ 2/10 & k = 1 \\ \dfrac{1}{10 - k} & 2 \leq k \leq 9 \end{cases}$$

and so it is not nondecreasing. However it does follow that N^* (the number sampled until a complete minimal path set is obtained) is increasing failure rate in this case. Hence we have

Corollary 2:

If the minimal path sets do not overlap then $P\{N^* = k + 1 \mid N^* > k\}$ is nondecreasing in k, or, equivalently,

$$P\{N = i \mid N < i + 1\} \text{ is nondecreasing in } i .$$

In the case of nonoverlapping minimal path sets of sizes $\underline{n} = (n_1, \ldots, n_r)$ let us define $P_i(\underline{n})$ as the probability that the i^{th} minimal path set is the first minimal path set that is completely sampled. This quantity $P_i(\underline{n})$ was extensively studied in [2] under the interpretation that n_i was the size of the i^{th} nonoverlapping minimal cut set (and so $P_i(\underline{n})$ could be thought of in [2] as the probability that system failure is "caused" by the i^{th} minimal cut set).

Now let us define $P_i^*(\underline{n})$ as the probability that the i^{th} minimal path set is the last minimal path set to have had a component sampled before the system failed (i.e., $P_i^*(\underline{n})$ is the probability that the component whose failure causes system failure is from the i^{th} minimal path set).

Again by using the duality of outcomes $(i_1, \ldots, i_n)$ and $(i_n, \ldots, i_1)$ we immediately see that

$$P_i^*(\underline{n}) = P_i(\underline{n}) .$$

(The reasoning is as follows: if all the components of the i^{th} minimal path set have just been sampled then at that moment the set of remaining components does not contain a component of the i^{th} minimal path set - however the set of remaining components would have contained a component of the i^{th} minimal path set one selection earlier.) Hence all the properties for $P_i(\underline{n})$ derived in [2] also hold for $P_i^*(\underline{n})$. For instance, if we hold n_1 fixed and think of $P_i^*(\underline{n})$ as a function of $(n_2, \ldots, n_r)$ then as shown in Theorem 4.8 of [2], this is a Schur concave function of $(n_2, \ldots, n_r)$.

Duality also yields additional results about N - namely writing $N(\underline{n})$ as a function of $\underline{n} = (n_1, \ldots, n_r)$ and using the equivalence between $N(\underline{n})$ and $N^*(\underline{n})$ it follows from the result about $N^*(\underline{n})$ given in Theorem 7.3 of [2] that

$$P\{N(\underline{n}) < \ell\} \text{ is Schur concave in } \underline{n} .$$

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